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# THE ALGEBRAIC STRUCTURE OF THE N-BODY PROBLEM

*by L. M. Rauch*

Prepared under Grant No. NsG-413 *by*
SETON HALL UNIVERSITY
South Orange, N. J.

*for*

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ABSTRACT

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The basic concern in this paper is to unfold the algebraic structure of the n-body problem. It is found that the union of initially disparate sub-groups leads to a group whose properties are the formal tools that formulate the explicit solutions of the n-body problem. A brief indication of the applications of this evolved algebra is given in the last section of the paper.

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## INTRODUCTION

The two known general processes in the solution of the N-body problem, namely, the serial method of Steffensen [1], [2], [3] and the iterative process of Rauch [4], [5] are implicit formulations. They lead only to numerical evaluations. To remedy this defect explicit solutions are necessary so that (among other things) the interpretations of the dynamical states, from the analytical phases, may be given directly other than through numerization.

In the attempt to obtain such solutions it was observed that more basic formal considerations were necessary to gain the desired end. These initial observations generated the present paper on the algebraic structure of the N-body problem.

The paper is divided into two parts. The first part deals with two collections of initially disparate elements and the union of these two sets. The elements of the first set are concrete, generated from an initial element which essentially specifies the canonical form of the equations of motion of the N bodies. The elements of the second aggregate are, at first, abstract but are given a rudimentary character through what appears to be an arbitrary definition. It is then found that the union set endows certain properties (embodied in theorems) on the initially abstract second subset which then allows us to deduce the group properties of the union ensemble.

These group properties, as well as more concrete formulations, are deduced in the second part of the paper. These concrete aspects are precisely the tools that formulate the explicit solutions (with their concomitant properties such as regions of convergence, singularities, etc.) of the problem.

The paper is, of course, concerned only with the formal aspects of the N-body problem, however, the last section of the second part gives a brief indication of the applications of the evolved algebra.

## I. THE ALGEBRAIC STRUCTURE

In this part the equations of motion are reduced to a symbolic canonical form. This symbolic representation suggests the group character of the set of elements which constitute not only an abstract solution of the N-body problem in application, but also its concrete formulation. The equations of motion are not regularized [6], [7] since the abstract structure of the set is thus more readily attained.

The symbolic form of the equations of motion. The position vector $x^i$ of the $i^{th}$ point mass $M_i$ is given by its components $x^{ih}$ relative to an inertial Cartesian rectangular coördinate frame. It is specified that the indices i, j, h, range over the positive integers in the form,

$$i,j = 1,2,---,n; \; h = 1,2,3; \; i \neq j \tag{1.1}$$

The equations of motion of the N-bodies, in a potential field V, are given in the Newtonian form,

$$\ddot{x}^{ih} = \frac{1}{M_i} \frac{V}{X^{ih}}, \; V = \sum_{i,j} \frac{G\, M_i\, M_j}{(R^{ij})^{1/2}} \tag{1.2}$$

where the scalar $R^{ij}$, the square of the magnitude of the position vector between the i and j particles, is given by

$$R^{ij} = \sum_h (x^{ih} - x^{jh})^2 \tag{1.3}$$

If the vector $X^{ij}$ is defined as

$$X^{ij} = X^i - X^j, \tag{1.4}$$

then its components are expressed by the symbol

$$X^{ij}(h) = x^{ih} - x^{jh} \tag{1.5}$$

and so

$$R^{ij} = \sum_h \left[X^{ij}(h)\right]^2 = R^{ji}. \tag{1.6}$$

With the further definition of the scalar

$$S^{ij} = \left(R^{ij}\right)^{-3/2}, \tag{1.7}$$

the equations of motion take the form

$$\ddot{x}^{ih} = \sum_j H_j S^{ij} X^{ij}(h); \; H_j = GM_j; \; i \neq j \tag{1.8}$$

Since the type of operations that follow leave the indices unaltered, we delete these and reintroduce them at appropriate points. The given system of differential equations thus formally become

$$\ddot{x} = \sum_j H_j S^j X^j = H_h S^j X^j \tag{1.9}$$

where the summation on the j is understood. The above symbolic form (1.9) may be further modified by the definitions

$$x_2 = HS_o X_o \; ; \; x_2 \equiv \ddot{x}, X_o \equiv X, S_o \equiv S \tag{1.10}$$

Define

$$y_o \equiv S_o X_o, \tag{1.11}$$

so that finally the equations of motions in the form (1.2) have been transformed into the canonical symbolic form

$$x_2 = H_{y_o} \tag{1.12}$$

The set of derivatives with positive subscripts.

The union R of the two sets $R_1$ and $R_2$ of elements $(y_o, y_1, y_2, ---)$ and $(---, y_{-3}, y_{-2}, y_{-1})$ respectively, will be shown to be a group under a well defined multiplicative operation. Some significant properties of this group will be formulated, among which will be the central fact that any element of the set, $R = R \cup R_2 = \{---, y_{-2}, y_{-1}, y_o, y_1, y_2, ---\}$, in the order indicated, will be the derivative (relative to time) of an immediate predecessor and the integral of an immediate successor.

In this section we deal only with the set $R_1$ of elements with positive subscripts or zero.

The elements $(y_o, y_1, y_2, ---) \in R_1$ as they stand now are wholly abstract (undefined) except for $y_o$ (specified by (1.11)). We proceed to express the elements $(y_1, y_2, y_3, ---)$ with positive subscripts by means of Leibnitz's rule, applied to $y_o$, for the determination of the $r^{th}$ derivative of the product of two functions $S_o(t)$ and $X_o(t)$. Apply the rule to definition (1.11). Thus

$$y_r \equiv (y_o)_r \equiv \frac{d^r y_o}{dt^r} = \sum_{p=o}^{r} \binom{r}{p} S_{r-p} X_p = \sum_{p=o}^{r} \binom{r}{p} X_{r-p} S_p ; r = o,1,2,---, \quad (2.1)$$

where the well known symbol $\binom{r}{p} = \dfrac{r!}{p!(r-p)!}$ represents the coefficients of a binomial expansion and where

$$S_{r-p} \equiv \frac{d^{r-p} S_o}{dt^{r-p}}, \quad X_p \equiv \frac{d^p X_o}{dt^p}, \quad r \geq p. \quad (2.2)$$

The elements $y_r \in R_1$, with positive integral subscripts or zero, may be viewed as now defined by means of (2.1) and (2.2). We will call these elements, for brevity, positive ones.

The property of closure on $R_1$, with a rule of multiplication for any two elements $y_r, y_m \in R_1$, is formulated by

Theorem 1 - The product of any two elements $y_r, y_m \varepsilon R_1$ is an element of $R_1$ given by

$$y_r y_m = y_m y_r = y_{r+m}; \quad r,m = o,1,2,--- \tag{2.3}$$

where the following conditions on the S's and X's hold:

$$S_\alpha S_\beta = S_{\alpha+\beta}, \quad X_\gamma X_\delta = X_{\gamma+\delta}; \quad \alpha_1 \beta, \gamma, \delta = o,1,2,--- \tag{2.4}$$

The proof is as follows. In view of Eq. (2.1)

$$y_{r+m} = \sum_{s=o}^{r+m} \binom{r+m}{s} S_{r+m-s} X_s, \quad r+m \geq o$$

By the well known addition rule for binomial coefficients [9], [10], namely

$$\binom{r+m}{s} = \sum_{q=o}^{s} \binom{m}{q} \binom{r}{s-q} \quad ; \ m \geq s, \ r \geq s, \tag{2.5}$$

it follows that

$$y_{r+m} = \sum_{s=o}^{r+m} \sum_{q=o}^{s} \binom{m}{q} \binom{r}{s-q} S_{r+m-s} X_s = \sum_{s=o}^{r+m} S_{r+m-s} X_s \sum_{q=o}^{s} \binom{m}{q} \binom{r}{s-q}. \tag{2.6}$$

For the interval $s+1 \leq q \leq m$, we have $s-q \leq -1$. If use is made of the combinatorial fact that

$$\binom{x}{r} = o, \text{ for any integer } r < o \text{ or } r > x \text{ and any } x \tag{2.7}$$

then

$$\sum_{q=s+1}^{m} \binom{m}{q}\binom{r}{s-q} = o, \text{ since } s-q \leq -1, \text{ namely, } s-q < o$$

Thus

$$\sum_{q=o}^{m} \binom{m}{q}\binom{r}{s-q} \equiv \sum_{q=o}^{s} \binom{m}{q}\binom{r}{s-q} + \sum_{q=s+1}^{m} \binom{m}{q}\binom{r}{s-q} = \sum_{q=o}^{s} \binom{m}{q}\binom{r}{s-q} .$$

Formula (2.6) thus becomes

$$y_{r+m} = \sum_{s=o}^{r+m} \sum_{q=o}^{m} \binom{m}{q}\binom{r}{s-q} S_{r+m-s} X_s$$

Use the substitution $p = s - q$ with the corresponding range for p; namely when $s = o$ and $q = o$, then $p = o$. Likewise when $s = r + m$ and $q = m$, then $p = r$. This leads to the expression

$$y_{r+m} = \sum_{p=o}^{r} \sum_{q=o}^{m} \binom{m}{q}\binom{r}{p} S_{r-p+m-q} X_{p+q}$$

If it is specified that the rules for the multiplication of the S's and the X's are to be the conditions given by (2.4) then

$$y_{r+m} = \sum_{p=o}^{r} \binom{r}{p} S_{r-p} X_p \sum_{q=o}^{m} \binom{m}{q} S_{m-q} X_q = y_r y_m ; r,m = o,1,2,---,$$

by definition (2.1).

The commutative rule is readily shown so that the theorem has been validated.

In particular for, say, $r = o$

$$y_o y_m = y_{o+m} = y_m, \text{ with}$$

$$S_o S_r = S_r, \; X_o X_n = X_n ; \; m_1 n_1 r \geq o \tag{2.8}$$

We may thus formulate the

Corollary - The element $y_o \in R_1$ is a multiplicative identity element of the set $R_1$.

It is of some interest to note that the set of elements generated by the binomial formula for non-negative integral powers,

$$(S + X)^r = \sum_{p=o}^{r} \binom{r}{p} S^{r-p}X^p, \ r = 0,1,2,--- \tag{2.9}$$

are isomorphic to the set $R_1$ of positive elements generated by the Leibnitz **formula** (2.1). For, briefly, if the powers are transformed to subscripts and if the S and X are defined as $S \equiv S_o$, and $X \equiv X_o$ and further, if $y_o \equiv (S + X)_o$, then (2.9) turns into (2.1), namely

$$y_r \equiv (S + X)_r = \sum_{p=o}^{r} \binom{r}{p} S_{r-p}X_p, \ r = 0,1,2,--- \tag{2.10}$$

and where, for r = 0

$$y_o \equiv (S + X)_o = S_oX_o. \tag{2.11}$$

Mixed set of elements - belonging to $R_1$ and $R_2$. The character of the elements $\in R_2$, namely, those elements with negative subscripts (negative elements) differ in the concrete basically from the positive ones ($\in R_1$) in their generation and mode of representation. They have no process of generation from an initial element $y_o$ unless we define arbitrarily an ad hoc principle of generation and secondly, as will be noted, their mode of representation is no longer in finite terms of the S's and X's.

Let us consider initially and briefly a heuristic point of view from which we can draw a provisional definition (through analogy) of any negative element. Thus consider the binomial expansion for negative integral powers,

$$\left(S + X\right)^{-m} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{m-1} S^{-m-r} X^r = \sum_{r=o}^{\infty} (-1)^r \frac{\Gamma(m+r)}{\Gamma(m)\Gamma(r+1)}$$

$$S^{-m-r} X^r; \; m = 0,1,2,--- \tag{3.1}$$

where the gamma functions are used in the last member in the equalities.

We now start ab initio and express any element $y_{-m} \varepsilon R_2$ in terms of the S's and X's, by means of an analogous formula,

$$\left(y_o\right)_{-m} \equiv y_{-m} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} \quad S_{-m-r} X_r = \sum_{r=0}^{\infty} (-1)^r \; \frac{\Gamma(m+r)}{\Gamma(m)\Gamma(r+1)}$$

$$S_{-m-r} X_r, \; m = 0,1,2,--- \tag{3.2}$$

(The case m = 0 is included to give the formula a trifle larger scope, though the value $y_o$ derived from (3.2) belongs, by definition, to the set $R_1$). The statement (3.2) for $y_{-m}$ is wholly abstract. No meaning can be attached to the element $y_{-m} \varepsilon R_2$ since no definition is explicit in (3.2), namely the unknown $y_{-m}$ is given in terms of a set of unknowns $\{S_{-m-r};\ 0 \leq r < \infty\}$. If the set of scalars $\{S_{-m-r}\}$ could be determined the N-body problem would likewise be resolved.

We will now attempt to give some character to the elements $y_{-m}$ (other than that found in the almost empty definition (3.2). In the previous section any two positive elements were combined (by a defined multiplicative operation) to form an element of $R_1$ . We consider now the second possible case in which $y_n \varepsilon R_1, y_{-m} \varepsilon R_2$ such that $y_{-m}\ y_n \varepsilon R$ where $R = R_1 U R_2$. We formulate closure on the set R by

<u>Theorem 2</u> - If $y_{-m} \varepsilon R_2$ and $y_n \ \varepsilon \ R_1$ then

$$y_{-m}y_n = y_{-m+n}\ ,\ y_{-m+n}\varepsilon\ R\ ;\ m,n = 0,1,2,--- \tag{3.3}$$

with the conditions that

$$S_{-\alpha}\ S_\beta = S_{-\alpha+\beta},\ X_{-\gamma}X_\delta = X_{-\gamma+\delta};\ \alpha,\beta,\gamma,\ \delta = 0,1,2,--- \tag{3.4}$$

The expansion of $y_{-m+n}$ is, with (3.2) in view, given as

$$y_{-m+n} = y_{-(m-n)} = \sum_{r=o}^{\infty} (-1)^r \binom{m-n+r-1}{r} S_{-(m-n)-r}X_r\ ;$$

$$m,n = 0,1,2,--- \tag{3.5}$$

It is to be observed that if $m - n < 0$ or $m < n$ (so that $y_{-(m-n)}\varepsilon R_1$), then expression (3.5) turns into a finite form of the type (2.1). We are thus concerned with the more involved case where $m - n > 0$ or $m > n$, namely with the situation where $y_{-(m-n)}\varepsilon R_2$.

The left member of (3.3) is given, by means of (3.2) and (2.1), in the form

$$y_{-m}y_n = \sum_{r=o}^{\infty}(-1)^r \binom{m+r-1}{r} S_{-m-r}X_r \sum_{s=o}^{n} \binom{n}{s} S_{n-s}X_s.$$

The factor $\sum_{s=o}^{n} \binom{n}{s} S_{n-s}\ X_s$ is transformed to an infinite sum by the following device. With the combinatorial formula (2.7) in mind,

$$\sum_{s=n+1}^{\infty} \binom{n}{s} S_{n-s}X_s = 0.$$

so that

$$\sum_{s=o}^{\infty}\binom{n}{s}S_{n-s}X_s=\sum_{s=o}^{n}\binom{n}{s}S_{n-s}X_s+\sum_{s=n+1}^{\infty}\binom{n}{s}S_{n-s}X_s=\sum_{s=o}^{n}\binom{n}{s}S_{n-s}X_s.$$

It follows that

$$y_{-m}y_n=\sum_{r=o}^{\infty}(-1)^r\binom{m+r-1}{r}S_{-m-r}\,X_r\sum_{s=o}^{\infty}\binom{n}{s}S_{n-s}X_s$$

Cauchy's formula for the product of two sums, namely

$$\sum_r u_r\sum_r w_r=\sum_r\ \sum_{t=o}^{r}u_{r-t}\,w_t,$$

is utilized. It follows that

$$y_{-m}y_n=\sum_{r=o}^{\infty}\sum_{t=o}^{r}(-1)^{r-t}\binom{m+r-t-1}{r-t}\binom{n}{t}S_{-m-r+t}X_{r-t}S_{n-t}X_{+t}.$$

If multiplication is defined for the S's and X's by the conditions that (3.6) $S_\alpha S_\beta = S_{\alpha\ \beta}$, $X_\gamma X_\delta = X_{\gamma+\delta}$ for all integers $\alpha,\beta,\gamma,\delta$ (3.6)

then

$$y_{-m}y_n=\sum_{r=o}^{\infty}(-1)^r\sum_{t=o}^{r}\binom{m+r-t+1}{r-t}S_{-m+n-r}X_r.\qquad(3.7)$$

If a comparison is made between (3.7) and (3.5), it is observed that the two left members will be identical provided the condition

$$\sum_{t=o}^{r}(-1)^t\binom{m+r-t-1}{r-t}\binom{n}{t}=\binom{m-n+t-1}{r}\qquad(3.8)$$

for any r, is satisfied. This must be shown.

The well known combinatorial form

$$\binom{-v}{w} = (-1)^w \binom{v+w-1}{w} \tag{3.9}$$

is utilized in what follows. Let $T = r - t$. So that when $t = 0$ then $T = r$ and when $t = r$, $T = 0$. These substitutions are used in the left member of (3.8). Thus

$$\sum_{t=o}^{r} (-1)^t \binom{m+r-t-1}{r-t}\binom{n}{t} = (-1)^r \sum_{T=o}^{r} (-1)^T \binom{m+T-1}{T}\binom{n}{r-T}$$

Use (3.9) from right to left, to get

$$\sum_{t=o}^{r} (-1)^t \binom{m+r-t}{r-t}\binom{n}{t} = (-1)^r \sum_{T=o}^{r} \binom{-m}{T}\binom{n}{r-T}.$$

If the addition formula (2.5) is used on the above right member,

$$\sum_{t=o}^{r} (-1)^t \binom{m+r-t}{r-t}\binom{n}{t} = (-1)^r \binom{-m+n}{r}.$$

Let $\binom{-m+n}{r} = \binom{-(m-n)}{r}$ and again use (3.9), then

$$\binom{-m+n}{r} = (-1)^r \binom{m-n+r-1}{r}$$

and from which we get

$$\sum_{t=o}^{r} (-1)^t \binom{m+r-t}{r-t}\binom{n}{t} = (-1)^{2r} \binom{m-n+r-1}{r} = \binom{m-n+r-1}{r}.$$

But this is precisely condition (3.8). Relations (3.3) are thus seen to be valid under conditions (3.4).

Elements belonging only to $R_2$

In the last two sections we dealt with the multiplicative rule for any two elements such that $y_m \epsilon R_1$, $y_n \epsilon R_1$ and $y_{-m} \epsilon R_2$, $y_n \epsilon R_1$, respectively.

However there exists no a priori reason that the third possibility must be fulfilled, namely that if $y_{-m} \epsilon R_2$, $y_{-n} \epsilon R_2$ then $y_{-m} y_{-n} \epsilon R_2$. We now formulate, in a theorem the closure property on $R_2$ for negative elements.

Theorem 3 - If $y_{-m}$, $y_{-n} \epsilon R_2$, then

$$y_{-m} y_{-n} = y_{-m-n}, \; y_{-m-n} \epsilon R_2; \; m,n = 0,1,2,--- \qquad (4.1)$$

with the conditions

that

$$S_{-\alpha} S_{-\beta} = S_{-(\alpha+\beta)}, \; X_{-\gamma} X_{-\delta} = X_{-(\gamma+\delta)}; \alpha,\beta,\gamma,\delta = 0,1,2,---. \qquad (4.2)$$

The element $y_{-m-n} \; \epsilon \; R_2$ is specified, by means of (3.2) as

$$y_{-m-n} = y_{-(m+n)} = \sum_{r=o}^{\infty} (-1)^r \binom{m+n+r-1}{r} S_{-(m+n)-r} X_r \; . \qquad (4.3)$$

The product of the two elements take the form

$$y_{-m} y_{-n} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} S_{-m-r} X_r \sum_{s=o}^{\infty} (-1)^s \binom{m+s-1}{s} S_{-n-s} X_s .$$

Use the Cauchy multiplicative formula,

$$y_{-m}y_{-n} = \sum_{r=o}^{\infty} \sum_{t=o}^{r} (-1)^{r-t} \binom{m+r-t-1}{r-t} S_{-m-r+t}X_{r-t}(-1)^{t}\binom{n+t-1}{t}$$

$$S_{-n-t}X_{t} = \sum_{r=o}^{\infty} (-1)^{r} \sum_{t=o}^{r} \binom{m+r-t-1}{r-t}\binom{n+t-1}{t} S_{-m-n-r}X_{r} ,$$

in view of the assumed conditions that

$$S_{-\alpha}S_{-\beta} = S_{-\alpha-\beta},\ X_{-\gamma}X_{-s} = X_{-\gamma-\delta};\ \alpha,\beta,\gamma,\delta = 0,1,2,---. \tag{4.4}$$

If $y_{-m-n} = y_{-m}y_{-n}$, then for every r in the interval $(0,\infty)$, we have

$$\sum_{t=o}^{r} \binom{m+r-t-1}{r-t}\binom{n+t-1}{t} \quad \binom{m+n+r-1}{r} . \tag{4.5}$$

To show the validity of (4.5), use the addition formula (2.5). Then

$$\sum_{t=o}^{r} \binom{m+r-t-1}{r-t}\binom{n+t-1}{t} = \binom{m+n+r-1-1}{r}$$

By means of (3.9)

$$\binom{n+t-1}{t} = (-1)^{t} \binom{-n}{t}.$$

So that

$$Q \equiv \sum_{t=o}^{r}\binom{m+r-t-1}{r-t}\binom{n+t-1}{t} = \sum_{t=o}^{r} \binom{m+r-t-1}{r-t} (-1)^{t}\binom{-n}{t}$$

Let $T = r - t$ , then

$$Q = (-1)^r \sum_{T=o}^{r} (-1)^T \binom{m+T-1}{T}\binom{-n}{r-T} .$$

Use (3.9) so that $(-1)^T \binom{m+T-1}{T} = \binom{-m}{T}$

and

$$Q = (-1)^r \sum_{T=o}^{r} \binom{-m}{T}\binom{-n}{r-T}.$$

In view of (2.5),

$$Q = (-1)^r \binom{-m-n}{r} = \binom{-(m+n)}{r}$$

Again use (3.9) to give the value

$$Q = \binom{m+n+r-1}{r}$$

or

$$\sum_{t=o}^{r} \binom{m+r-t-1}{r-t} \binom{n+t-1}{t} = \binom{m+n+r-1}{r}$$

But the validity of this (expression (4.5)) was necessary to prove the theorem.

<u>Generalization of the multiplicative formulae.</u> The three theorems of closure for the elements $y_m$, $y_n \varepsilon R_1$; $y_m \varepsilon R_1$, $y_n \varepsilon R_2$; $y_m$, $y_n \varepsilon R_2$ under multiplication may be incorporated into a single theorem. Before doing this we will show that a single formula can express any element $y_m \varepsilon R$. This fact is given by

Theorem 4 - Any element $y_m \varepsilon$ R is given by the expansion

$$y_m = \sum_{r=0}^{\infty} \binom{m}{r} S_{m-r} X_r \qquad \text{(m any integer)} \tag{5.1}$$

Consider any negative integer or zero,

$m = - t$, $t = 0, 1, 2,$ ---.

Expression (5.1) then becomes

$$y_{-t} = \sum_{r=0}^{\infty} \binom{-t}{r} S_{-t-r} X_r \tag{5.2}$$

On the basis of (3.9) we can write that $\binom{-t}{r} = (-1)^r \binom{t+r-1}{r}$,

so that (5.2) becomes

$$y_{-t} = \sum_{r=0}^{\infty} (-1)^r \binom{t+r-1}{r} S_{-t-r} X_r \ , \ t = 0,1,2,---. \tag{5.3}$$

But that is the form (3.2) for any element $\varepsilon$ $R_2$.

Now let m be a positive integer or zero, namely $m = n$, $n = 0, 1, 2,$ ---. Then (5.1) becomes

$$y_s = \sum_{r=0}^{\infty} \binom{s}{r} S_{s-r} X_r \tag{5.4}$$

Now

$$\sum_{r=0}^{\infty} \binom{s}{r} S_{s-r} X_r = \sum_{r=0}^{s} \binom{s}{r} S_{s-r} X_r + \sum_{r=s+1}^{\infty} \binom{s}{r} S_{s-r} X_r \ .$$

But by the combinatorial formula (2.7), $\binom{s}{r} = 0$ for $r > s$.

So that $\sum_{r=s+1}^{\infty} \binom{s}{r} S_{s-r} X_r = 0$ and (5.4) changes to

$$y_s = \sum_{r=0}^{s} \binom{s}{r} S_{s-r} X_r, \ s = 0,1,2,---, \tag{5.5}$$

which is the form (2.1). Thus Theorem 4 is valid.

We now consider the general closure principle given by

<u>Theorem</u> 5 - If $y_m, y_n \in R$, then

$$y_m y_n = y_{m+n}, \; y_{m+n} \in R \text{ for any integer, } m,n, \qquad (5.6)$$

with the conditions

$$S_\alpha S_\beta = S_{\alpha+\beta}, \; X_\gamma X_\delta = X_{\gamma+\delta} \quad \text{for any integer, } m,n. \qquad (5.6')$$

The proof is readily achieved with the use of expression (5.1). The element $y_{m+n}$ is given by

$$y_{m+n} = \sum_{r=0}^{\infty} \binom{m+n}{r} S_{m+n-r} X_r \text{ for any integer } m,n. \qquad (5.7)$$

Use (2.5) to get

$$y_{m+n} = \sum_{r=0}^{\infty} \binom{n}{t} \binom{m}{r-t} S_{m+n-r} X_r \text{ for any integer } m,n. \qquad (5.8)$$

Develop $y_m y_n$ in the form

$$y_m y_n = \sum_{r=0}^{\infty} \binom{m}{r} S_{m-r} X_r \sum_{s=0}^{\infty} \binom{n}{s} S_{n-s} X_s .$$

Cauchy's multiplication formula leads to

$$y_m y_n = \sum_{r=0}^{\infty} \sum_{t=0}^{r} \binom{m}{r-t} S_{m-r+t} X_{r-t} \binom{n}{t} S_{n-t} X_t .$$

If the conditions (5.7) are satisfied,

$$y_m y_n = \sum_{r=o}^{\infty} \sum_{t=o}^{r} \binom{m}{r-t} \binom{n}{t} S_{m+n-r} X_r \text{ for any integer } m,n. \quad (5.9)$$

But (5.9) is precisely (5.7). The theorem is thus proved.

The coefficients $\binom{m}{r}$ of (5.1) may be put in different functional forms. Two other forms are considered here. In terms of gamma functions

$$\binom{m}{r} = \frac{m!}{(m-r)!r!} = \frac{\Gamma(m+1)}{\Gamma(m-r+1)\Gamma(r+1)}$$

Though the gamma functions are undefined for negative integral arguments or zero yet the total expression is defined. We thus have a second form for (5.1), namely

$$y_m = \sum_{r=o}^{\infty} \frac{\Gamma(m+1)}{\Gamma(r+1)\Gamma(m-r+1)} S_{m-r} X_r \text{ (m any integral} \quad (5.10)$$

value or zero.)

From the form (5.10) another expression may be formulated in terms of the hypergeometric functions [11.]. Consider the well known relation between the gamma and hypergeometric functions, namely

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = F(a,b;c;1) \quad (5.11)$$

where $c \bullet a \bullet b > 0$ and where the quantities are assumed real.

Specify the identity

$$\frac{\Gamma(m\ 1)\,\Gamma\,(x)}{\Gamma(r+1)\Gamma(m-r+1)} \equiv \frac{\Gamma(c)\,\Gamma\,(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

where $\Gamma(x)$ is introduced with the unknown argument and to be determined with the quantities a,b,c by the four linear equations derived from the identity, namely

$$c = m+1,\ c-a-b = x,\ c-a = r+1,\ c-b = m-r+1.$$

The above equations lead to the values

$$a = m-r,\ b = r,\ c = m+1,\ x = 1$$

It further follows that, since $c-a-b = x = 1 > 0$, the condition imposed on the relation (5.11) is satisfied. So that, since $\Gamma(x) = \Gamma(1) = 1$

$$\frac{\Gamma(m+1)}{\Gamma(r+1)\Gamma(m-r+1)} = F(m-r,r;m+1;1)$$

Thus a third form for the expression of the general formula (5.1) is given as

$$y_m = \sum_{r=o}^{\infty} F\,(m-r,r;m+1;1)S_{m-r}X_r \quad \text{(m any integer)} \tag{5.12}$$

For the purpose of reference and to observe the concrete form which is the base of the general formulae (5.1), (5.10) or (5.12) the indices are re-inserted. If we bear the equations of motion (1.12) in mind, then

$$X_{m+2} = H\ y_m \tag{5.13}$$

or

$$X_{m+2}^{ih} = \sum_j H_j y_m^{ih} \tag{5.13'}$$

Use formulae (5.1), (5.10), (5.12). So that

$$x_{m+2}^{ih} = \sum_j \sum_{r=o}^{\infty} H_j \binom{m}{r} S_{m-r}^{ij}\ X^{ij}(h) = \sum_j \sum_{r=o}^{\infty} H_j \frac{\Gamma(m+1)}{\Gamma(r+1)\Gamma(m-r+1)}$$

$$S_{m-r}^{ij}\ X_r^{ij}(h) = \sum_j \sum_{r=o}^{\infty} H_j F(m-r,r;m+1;1)S_{m-r}^{ij}X_r^{ij}(h). \tag{5.14}$$

## II. THE BASIC PROPERTIES OF THE SET R AS A GROUP

The previous part of the paper, was concerned mainly with the formulation of the properties of the set $R = R_1 U R_2$ from a well defined generating element $y_o \varepsilon R_1$, derived from the canonical form of the equations of motion of the N bodies. Though the elements $(---, y_{-3}, y_{-2}, y_{-1})$ $\varepsilon R_2$, unlike the elements $(y_1, y_2, y_3, ---) \varepsilon R_1$, were initially undefined yet (by what at first appeared to be an arbitrary or at best a heuristic definition (3.2)), these negative elements were incorporated into the collection R. The principle of closure thus gave properties to the set of elements $R_2$, those properties that belong to the total aggregate R.

Our object in part II of the paper is (1) to unfold some of the properties of R and (2) to indicate, very broadly and briefly, a few processes (involving the properties of R) by means of which the N-body problem may be resolved explicitly.

<u>The set $R = R_1 U R_2$ as a group</u> - The first property that we establish for the set R is that it is a group under the defined multiplicative operation. This property was not explicitly given in the previous discussion. Here we emphasize those facts which determine that a collection of elements constitutes a group [12].

<u>Theorem 6</u> - The set $R = R_1 U R_2$ is an Abelian group of infinite order, where

$$y_m \ \varepsilon \ R_1, \ m = 0,1,2,--- \ ; \ y_{-n} \ \varepsilon \ R_2, \ n = 1,2,3,--- \tag{6.1}$$

$$y_m = \sum_{p=o}^{m} \binom{m}{p} S_{m-p} X_p ; \ y_{-n} = \sum_{p=o}^{\infty} \binom{n+p-1}{p} S_{-n-p} X_p \tag{6.2}$$

$$y_{r+s} = y_r y_s, \ y_r y_s \ \varepsilon \ R \ ; \ r,s = 0, \pm 1, \pm 2, ---, \tag{6.3}$$

with the multiplicative conditions

$$S_\alpha S_\beta = S_{\alpha + \beta},\ X_\gamma X_\delta = X_{\gamma + \delta};\ \alpha,\beta,\gamma,\delta = 0, \pm 1, \pm 2,---;$$

(the scalar quantities S and X belong to the field of real or complex numbers)

(1) The element $y_o$ is the identity element of the set R, namely $y_o y_m = y_m y_o = y_m$ for the set of all integers $\{m\}$ . If the general form (5.1) is used, then

$$y_o y_m = S_o X_o \sum_{r=o}^{\infty} \binom{m}{r} S_{m-r} X_r = \sum_{r=o}^{\infty} \binom{m}{r} S_o X_o S_{m-r} X_r \quad \text{for any}$$

integer m

Use the multiplicative conditions (4), so that

$$y_o y_m = \sum_{r=o}^{\infty} \binom{m}{r} S_{m-r} X_r = y_m$$

The same result is obtained for $y_o y_o$.

(2) For every subscript m of $y_m$ there exists an integer $_{-m}$ of $y_{-m}$ such that

$$y_m y_{-m} = y_{-m} y_m = y_o.$$

To prove the existence of an inverse element $y_{-m} \in R$ corresponding to any element $y_m \in R$, the closure theorem 2 on mixed indices is used, namely, if $y_m \in R_1$, $y_{-m} \in R_2$,

The same theorem leads to the fact that $y_{-m} y_m = y_o$

(3) The associative rule for any three elements is valid, namely

$$(y_m y_n) y_r = y_m (y_n y_r).$$

This statement is immediately seen to be true by the general closure principle for multiplication. Thus

$$(y_m y_n) y_r = y_{(m+n)} y_r = y_{(m+n)+r} = y_{m+(n+r)} = y_m (y_n y_r).$$

(4) The set R is commutative. This is readily observed. If $y_m$, $y_n \varepsilon$ R, then by the general closure formula (5.6), $y_m y_n = y_{m+n}$, for all integers m, n.

However, in view of (5.1),

$$y_m y_n = y_{m+n} = \sum_{r=o}^{\infty} \binom{m+n}{r} S_{m+n-r} X_r = \sum_{r=o}^{\infty} \binom{n+m}{r} S_{n+m-r} X_r$$

$$= y_{n+m} = y_n y_m.$$

(5) The number of elements in R is a discrete infinity. This is manifest since the number in the set of all integers is a countable infinity.

The validity of these five properties of the set R establishes the validity of the theorem. If might be observed that since the set R, under the defined multiplicative operation, is isomorphic to the set of all integers under addition, the proof of the theorem could be more elegently established by noting this isomorphism.

<u>The basic non=algebraic property of the group R</u> - The essential meaning of the development of the algebraic structure of the set $R = R_1 U R_2$ relative to the solution of the N-body problem, lies in the implications of the non-algebraic property of the group R as given by

<u>Theorem 7</u> - If the elements in the group R are ordered in a sequence of increasing subscripts, then any element of the set is the derivative (relative to time) of it's immediate predecessor and is the integral (but for an additive constant) of its immediate successor.

For positive subscripts the theorem is immediately established by virtue of the rule of Leibnitz for the product of any two elements of $R_1$ . The rule is an inductive one such that, given the initial element $y_o \varepsilon R_1$ any positive element in the sequence is defined by an appropriate number of differentiations of an intially given element of the positive sequence. This is specifically so for the immediate precedessor of a specified element. If further the ordinary notion of integration is adhered to (as an operation inverse to differentiation), then any given element is the integral (but for an arbitrary constant) of an immediate successor in the ordered positive set ($R_1$).

However this inductive rule no longer necessarily applies for the total sequence R since the rule itself is established from the properties of the positive elements $(y_1, y_2, y_3, ---)$ and the assumption that $y_1$ is to follow $y_o$.

A more concrete formulation is given in the proof of the theorem for the positive sequence, $R_1$, by disregarding the Leibnitz inductive formula and making formulae (2.1) and (2.4) as ab initio definitions. Then since

$$y_r = \sum_{p=o}^{r} \binom{r}{p} S_{r-p}X_p, \quad r = 0,1,2,---, \tag{7.1}$$

$$\frac{dy_r}{dt} = \sum_{p=o}^{r} \binom{r}{p} \left[S_{r-p+1}X_p + X_{r-p}X_{p+1}\right]$$

By the use of (2.4),

$$\frac{dy_r}{dt} = \sum_{p=o}^{r} \binom{r}{p} S_{r-p}X_p\left(S_1X_o + S_oX_1\right) = \left(S_1X_o + S_oX_1\right)\sum_{p=o}^{r}\binom{r}{p}S_{r-p}X_p.$$

However since $y_1 = S_1X_o + S_oX_1$ and $y_r$ is given by (7.1), it follows that

$$\frac{dy_r}{dt} = y_1y_r = y_{r+1} \qquad \text{(because of (2.3))}$$

Further, by the definition of indefinite integration, namely that

$y_r = \int y_{r+1}dt + c$ (c, for the time being, is assumed zero), the theorem for the ordered set $R_1$ is established. (The element $y_o$ is assumed given, formula (1.12)).

The case for the sequence of elements belonging to $R_2$ is initially categorically different. The element $y_{-m} \in R_2$, $m > 0$ has initially no concrete meaning since it is not generated from $y_o$ inductively but is abstractly defined in terms of undefined symbols through the formula (3.2). We now show that the sequence of elements of $R_2$ have the same basic property (as given by the theorem) as the ordered elements of $R_1$.

The derivative of

$$y_{-m} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} S_{-m-r} X_r, \tag{7.2}$$

is given by

$$\frac{dy_{-m}}{dt} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} \left[S_{-m-r} X_{r+1} + S_{-m-r+1} X_r\right] \tag{7.3}$$

With the use of (3.4)

$$\frac{dy_{-m}}{dt} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} \left[S_o X_1 + S_1 X_o\right] S_{-m-r} X_r$$

$$= \left(S_o X_1 + S_1 X_o\right) \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} S_{-m-r} X_r$$

Since $y_1 = S_o X_1 + S_1 X_o$ and $y_{-m}$ is given by (7.2), it follows that

$$\frac{dy_{-m}}{dt} = y_1 y_{-m} = y_{-m+1}$$

With the added note on the definition of the indefinite integral (with c = 0), the theorem is shown to be also valid for the set of ordered elements of $R_2$. We make one manifest observation on the application to the N-body problem, namely, if the unknown quantities $S_{-q}$, $q = 1,2,3,---$ can be determined, then the N-body problem is resolved. We will give a brief elaboration on this point in the final section of the paper.

A division algorithm. The existence of an inverse element $y_{-m} \varepsilon R$ for any $y_m \varepsilon R$ namely $y_m y_{-m} = y_{-m} y_m = y_o$ leads to the question as to the meaning of a division symbolism, $\frac{y_o}{y_m}$. The answer is given by the

Theorem 8 - The division symbol $\frac{y_o}{y_m}$ means $y_{-m}$ if the process of division is carried through such that the multiplicative conditions

$$S_\alpha S_\beta = S_{\alpha+\beta},\ X_\gamma X_\delta = X_{\gamma+\delta};\ \alpha,\beta,\gamma,\delta = 0,\ \pm 1,\ \pm 2,\ --- \tag{8.1}$$

are fulfilled.

Consider first an example to illustrate a concrete situation in the division algorithm in which the conditions (8.1) are involved. It is proposed to find the value of $\frac{y_o}{y_m}$ for m = 2.

Thus

$$\frac{y_o}{y_2} = \frac{S_o X_o}{S_2 X_o + 2S_1 X_1 + S_o X_2}.$$

Now carry through the indicated process with (8.1) in mind,

$$
\begin{array}{r|l}
 & S_{-2}X_o - 2S_{-3}X_1 + 3S_{-4}X_2 - \text{---} \\
\hline
S_2X_o + 2S_1X_1 + S_oX_2 & S_oX_o \\
 & \underline{S_oX_o + 2S_{-1}X_1 + S_{-2}X_2} \\
 & \quad -2S_{-1}X_1 - S_{-2}X_2 \\
 & \quad \underline{-2S_{-1}X_1 - 4S_{-2}X_2 - 2S_{-3}X_3} \\
 & \qquad 3S_{-2}X_2 + 2S_{-3}X_3 \\
 & \qquad \underline{3S_{-2}X_2 + 6S_{-3}X_3 + 3S_{-4}X_4} \\
 & \qquad - - - - - - - - - -
\end{array}
$$

If the process is continued, we find that

$$\frac{y_o}{y_2} = \sum_{r=o}^{\infty} (-1)^r \binom{r+1}{r} S_{-2-r}X_r .$$

But in view of (3.2) $\sum_{r=o}^{\infty} (-1)^r \binom{r+1}{r} S_{-2-r}X_r = y_{-2}$ holds.

Hence

$$\frac{y_o}{y_2} = y_{-2}.$$

In general after the division is performed under (8.1) for the given quantities, $\frac{y_o}{y_m}$ has the expression

$$\frac{y_o}{y_m} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} S_{-m-r}X_r = y_{-m} \tag{8.2}$$

This expression in the form of a theorem may be established by the inductive process. However, a somewhat simpler proof is given by the following consideration.

Start with the given fact that an inverse exists for each element $y_m \in R$, namely,

$$y_m y_{-m} = y_o$$

Now symbolically divide both members of the equation by $y_m$ so that

$$\frac{y_m}{y_m} y_{-m} = \frac{y_o}{y_m}$$

Perform the division algorithm on $\frac{y_m}{y_m}$ with (8.1) in view. This leads to the quantity $\frac{y_m}{y_m} = S_o X_o = y_o$.

So that $\frac{y_o}{y_m} = y_o y_{-m}$. But since $y_o y_{-m} = y_{-m}$, we have that $y_{-m} = \frac{y_o}{y_m}$ and thus establishing the theorem.

We here make the manifest observation that the theorem has formal validity only since the division expansion is expressed terms of the still unknown entities, the negative S's ($S_{-q}$, $q = 1,2,3,---$).

Not until $S_{-q}$ is evaluated can we ascribe any other but a formal meaning to $y_{-m} = \frac{y_o}{y_m}$. However, this paper can demand no more than the formal or structural phases in its theorems relative to N-body problem. A work in progress at present on the explicit solution of the N-body problem utilizes the above theorem in a significant analytical mode.

The division theorem may be generalized by posing a question whose answer is the generalization. Given a linear equation $y_m y_r = y_s$, what is the solution, $y_m$, when $y_r$ and $y_s$ are given? The solution is embodied in the

Corollary - If $y_r$, $y_s \varepsilon R$ (r,s any two integers) and $y_m y_r = y_s$ then $y_m = y_{s-r} \varepsilon R$

Divide both sides of the equation formally by $y_r$.

Thus $\frac{y_r}{y_r} y_m = \frac{y_s}{y_r}$. The factor $\frac{y_r}{y_r}$ of the left member becomes, through the division algorithm, $\frac{y_r}{y_r} = S_o X_o \equiv y_o$ (1.11). So that $\frac{y_r}{y_r} y_m = y_o y_m = y_m$. The right member

$\frac{y_s}{y_r}$ may be put in the form $\frac{y_s}{y_r} = \frac{y_o y_s}{y_r} = \frac{y_o}{y_r} y_s$.

But $\frac{y_o}{y_r} = y_{-r}$ by the division theorem and so $\frac{y_s}{y_r} = y_{-r} y_s$.

However, because of (3.3), $y_{-r}\, y_s = y_{s-r}$. Join the altered expressions for the left and right members of

$\frac{y_r}{y_r} y_m = \frac{y_s}{y_r}$ to get $y_m = y_{s-r}$. Thus the corollary is is shown to be valid with the observation that $y_{s-r} \varepsilon$ R.

## Brief outline of the application of the algebraic structure to the analytical solution of the N-body problem [13.], [8.].

The algebraic processes thus far developed suggests a number of modes of utilizing the formal structure in the explicit solution of the N-body problem. Since the explicit solution implies analytical considerations, we **must forego** detailed developments in this paper and deal **briefly** with the direction in which the formalism points.

Let us utilize a modified form of the basic Theorem 7 on the non-algebraic operations on the Group R. This modification creates a new set of elements, $\bar{R}$. Thus consider an arbitrary polynomial in t of given degree m - 1, namely $\sum_{i=o}^{m-1} C_i t^i$ ($C_i$, an arbitrary real quantity). The new element, $\bar{y}_{-m} \epsilon \bar{R}$ is given by the transformation

$$\bar{y}_{-m} = y_{-m} + \sum_{i=o}^{m-1} C_i t^i \text{ , } m = 1,2,3,---. \tag{9.1}$$

We note the obvious meaning of this transformation. For if we take a sequence of m derivatives on $\bar{y}_{-m}$, it is transformed into $\bar{y}_o = y_o$. This element, $y_o$, is, as we know, not only the identity of the group R but is the generic element from which the positive and the negative elements of R are generated by differentiation and its inverse respectively.

At least two processes for the **solution** of the N-body problems are suggested by the algebraic structure, thus far developed. We consider briefly two of these modes.

(1) The statement (3.2) modified, by the use of (9.1), leads to the form

$$\bar{y}_{-m} = \sum_{r=o}^{\infty} (-1)^r \binom{m+r-1}{r} S_{-m-r} X_r + \sum_{i=o}^{m-1} C_i t^i \tag{9.2}$$

It becomes known on the condition that $S_{-m-r}$, for any positive integer m > 0, is known. It is conjectured that the evaluation of an infinite determinant (shown to be convergent) will lead to known values of $S_{-m-r}$. If these values become known then their substitution in the equation (9.2) for m = 1 and m = 2, namely

$$\bar{y}_{-1} = \sum_{r=o}^{\infty} (-1)^r S_{-1-r} X_r + C_o ;$$

$$\bar{y}_{-2} = \sum_{r=o}^{\infty} (-1)^r (r+1) S_{-2-r} X_r + C_o + C_1 t \tag{9.3}$$

gives the solution of the N-body problem.

If we revert to the indices, then

$$\overline{X}^{ih}_{-m+2} = \sum_j H_j \overline{y}^{ih}_{-m} \tag{9.4}$$

by definition, and is consistent with (5.13'). If we utilize (5.14) and (9.2), then

$$\overline{X}^{ih}_{-m+2} = \sum_{j=1}^{n} H_j\left(y^{ih}_{-m} + \sum_{i=o}^{m-1} C_i t^i\right)$$

$$= \sum_{j=1}^{n} H_j y^{ih}_{-m} + \sum_{i=o}^{m-1} \sum_{j=1}^{n} H_j C_i t^i$$

But since $\sum_{j=1}^{n} H_j C_i$ is an arbitrary constant, say, $C_i$,

then

$$\overline{X}^{ih}_{-m+2} = \sum_{j=1}^{n} H_j y^{ih} + \sum_{i=o}^{m-1} C_i t^i \tag{9.5}$$

Formula (5.14) transforms (9.5) into

$$\overline{X}^{ih}_{-m+2} = \sum_{r=o}^{\infty} \sum_{j=1}^{n} H_j (-1)^r \binom{m+r-1}{r} S^{ij}_{-m-r} X^{ij}_r (h) + \sum_{i=o}^{m-1} C_i t^i. \tag{9.6}$$

Specifically for the solution of the N-body problem, m = 1,2, so that

$$\overline{X}_1{}^{ih} = \sum_{r=o}^{\infty} \sum_{j=1}^{n} H_j (-1)^r S^{ij}_{-1-r} X^{ij}_r (h) + C_o \tag{9.7}$$

$$\bar{X}_o^{ih} = \sum_{r=o}^{\infty} \sum_{j=1}^{n} H_j (-1)^r (r+1)\ S^{ij}_{-2-r} X^{ij}_r (h) + C_o + C_1 t \qquad (9.8)$$

These solutions are at the present formal. However, if the negative S's become known, the expressions (9.7), (9.8) become explicitly analytical solutions of the N-body problem. We leave this phase of the structural application for future analytical considerations.

An observation of some interest is to utilize the symmetrical character of the S's and the X's in the formulations. A manifest generalization of (2.1) namely that

$$y_r = \sum_{p=o}^{\infty} \binom{r}{p} S_{r-p} X_p = \sum_{p=o}^{\infty} \binom{r}{p} X_{r-p} S_r,$$

for any integer r, specifies this symmetry. So that if the S's and the X's are interchanged, equations (9.6), (9.7), (9.8) become

$$\bar{X}_{-m+2}^{ih} = \sum_{r=o}^{\infty} \sum_{j=1}^{n} H_j (-1)^r \binom{m+r-1}{r} X^{ij}_{-m-r}(h) S^{ij}_r + \sum_{i=o}^{m-1} C_i t^i, \qquad (9.9)$$

$$\bar{X}_1^{ih} = \sum_{r=o}^{\infty} \sum_{j=1}^{n} H_j (-1)^r\ X^{ij}_{-1-r}(h)\ S^{ij}_r + C_o. \qquad (9.10)$$

$$\bar{X}_2^{ih} = \sum_{r=o}^{\infty} \sum_{j=1}^{n} H_j (-1)^r (r+1)\ X^{ij}_{-2-r}(h) S^{ij}_r + C_o + C_i t^2. \qquad (9.11)$$

It is more likely that these expressions involving the unknown negative X's will turn out to be the more useful forms in future analytical developments.

(2) The second process in the analytical solution of the N-body problem and in which the evolved algebraic structure is utilized, involves the expansion of $y_{-2}$ (and $y_{-1}$) by a power series in t about, say, t = 0. Then by means of the

group properties of R, the non-algebraic operations of the group, the division theorem and its corollary, the analytical and the dynamical properties of the N-body physical system **may** be determined.

Thus consider the more general expansion, for any m,

$$y_{-m} = \sum_{r=-m}^{\infty} \frac{y_r(0)}{(r+m)!} t^{r+m} = \sum_{r=-m}^{-1} \frac{y_r(0)}{(r+m)!} t^{r+m}$$

$$+ \sum_{r=o}^{\infty} \frac{y_r(0)}{(r+m)!} t^{r+m}, \quad m = 1,2,3,- - - \tag{9.12}$$

where the first m quantities $(y_r(t))_{r=o} \equiv y_r(0)$; $r = - m, ---, - 1$ are given as initial conditions of the physical system.

Let $r = - s$, then (9.12) takes the form

$$y_{-m} = \sum_{s=1}^{m} \frac{y_{-s}(0)}{(m-s)!} t^{m-s} + \sum_{r=o}^{\infty} \frac{y_r(0)}{(r+m)!} t^{r+m}, \quad m = 1,2,3---. \tag{9.13}$$

Apply formula (2.1) to (9.13), to get

$$y_{-m} = \sum_{s=1}^{m} \frac{y_{-s}(0)}{(m-s)!} t^{m-s} + \sum_{r=o}^{\infty} \sum_{p=o}^{r} \frac{'}{(r+m)!} \binom{r}{p} \left[S_{r-p} X_p\right]_{t=o} t^{r+m} \tag{9.14}$$

Formula (9.14) is essentially, except for considerable detailed development, the solution of the N-body problem (for m = 1 and 2). Thus the expressions

$$y_{-1} = y_{-1}(0) + \sum_{r=o}^{\infty} \sum_{p=o}^{r} \frac{'}{(r+1)!} \binom{r}{p} \left(S_{r-p} X_p\right)_o t^{r+1},$$

$$y_{-2} = y_{-1}(0) + y_{-2}\ (0)\ t + \sum_{r=o}^{\infty} \sum_{p=o}^{r} \frac{'}{(r+\ )!} \binom{r}{p} \left( S_{r-p} X_p \right)_o t^{r+2} \qquad (9.16)$$

when fully evolved will specify the positions and velocities of the N-bodies relative to an inertial coordinate system, when the initial boundary conditions are given. One of the details in the development of the analysis and its dynamical interpretation is the determination of the region of convergence for the series and the consideration of the movable singularities of the solution [13.], [14.]. It will be found, as an example of the utilitarian aspect of the formal algebra, that the corollary to the division theorem points to such evaluation.

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